1-forms over the moduli space of irreducible connections defined by the spectrum of Dirac operators

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Abstract. *Let $\mathcal{M}(P)$ be the moduli space of irreducible connections of a G-principal bundle P over a closed Riemannian spin manifold M. Let D_A be the Dirac operator of M coupled to a connection A of P and f a smooth function on M. We consider a smooth variation A(u) of A with tangent vector ω and denote* $T_\omega := \frac{d}{du}(D_{A(u)} - f)\big|_{u=0}$. *The coefficients of the asymptotic expansion of trace* $(T_\omega \cdot e^{-t(D_A - f)^2})$ *near* $t = 0$ *define 1-forms* $\mathbf{a}_f^{(k)}$, $k = 0, 1, 2, \ldots$ *on* $\mathcal{M}(P)$. *In this paper we calculate* $\mathbf{a}_f^{(0)}, \mathbf{a}_f^{(1)}, \mathbf{a}_f^{(2)}$ *and study some of their properties. For instance using the 1-form* $\mathbf{a}_f^{(2)}$ *for suitable functions f we obtain a foliation of codimension 5 of the space of G-instantons of* S^4.

Let M^n be a closed Riemannian spin manifold, $n \geqslant 3$, G be a compact connected simple Lie group with Lie algebra g and P be a principal G-bundle over M^n. Let us denote by $\mathscr{C}^+(P)$ the space of all irreducible connections of P and by $\mathcal{M}(P) = \mathscr{C}^+(P)/\mathscr{G}(P)$ the orbit space with respect to the action of the gauge group $\mathscr{G}(P)$. Let $\pi : \mathscr{C}^+(P) \to \mathcal{M}(P)$ be the canonical projection. In the case $n = 4$ we denote by $\mathscr{A}^+(P)$ the space of all irreducible self-dual connections and by $\mathcal{N}(P) = \mathscr{A}^+(P)/\mathscr{G}(P)$ its orbit space.

Key-Words: Moduli spaces of instantons, asymptotic expansions, Dirac operators.
1980 MSC: 32G, 58G25.

In this article we will define for each function $f \in C^\infty(M^n)$ 1-forms $\mathbf{a}_f^{(k)}$ on $\mathcal{M}(P)$, $k = 0, 1, 2, \ldots$, arising from the coefficients of the asymptotic expansion of the heat kernel of certain Dirac-type operator, and study some of their properties.

By S we denote the spinor bundle of M^n to a fixed spinor structure.

Let $E = P \times_\rho V$ denote the vector bundle associated to P by a unitary representation $\rho : G \to U(V)$ and $\mathfrak{g} = P \times_{Ad} g$ the bundle associated by the adjoint representation of G. Let ∇^S be the spinor derivative and ∇^A be the derivative in $\Gamma(E)$ defined by a connection $A \in \mathscr{C}^+(P)$. Then the operator

$$D_A : \Gamma(S \otimes E) \to \Gamma(S \otimes E)$$

(1) $$\varphi \otimes e \to \sum_{j=1}^{n} (s_j \cdot \nabla^S_{s_j} \varphi \otimes e + s_j \cdot \varphi \otimes \nabla^A_{s_j} e)$$

$$(s_1, \ldots, s_n) \text{ local } ON\text{-basis}$$

is the Dirac operator on M^n coupled to a connection $A \in \mathscr{C}^+(P)$. For each function $f \in C^\infty(M^n)$ the operator $D_A - f$ is a 1^{st} order elliptic formal self-adjoint differential operator. $\mathscr{C}^+(P)$ is an affine space, whose vector space is the space $\Gamma(\mathfrak{g} \otimes \Lambda^1 M)$ of all 1-forms with values in the bundle $\mathfrak{g}$. For a smooth variation $A(u)$ of $A \in \mathscr{C}^+(P)$ with tangent vector $\omega \in \Gamma(\mathfrak{g} \otimes \Lambda^1 M)$ we consider the smooth family of differential operators $D_{A(u)} - f$ and denote by T_ω the derivative $T_\omega = \frac{d}{du}(D_{A(u)} - f)|_{u=0}$. Then (1) implies

$$T_\omega : \Gamma(S \otimes E) \to \Gamma(S \otimes E)$$

(2) $$\varphi \otimes e \to \sum_{j=1}^{n} s_j \circ \varphi \otimes \rho_*(\omega(s_j))\, e.$$

The operator $T_\omega \cdot e^{-t(D_A - f)^2}$ is an operator of trace class ($t > 0$). Trace $(T_\omega \cdot e^{-t(D_A - f)^2})$ has neat $t = 0$ an asymptotic expansion

$$\mathrm{Trace}(T_\omega e^{-t(D_A - f)^2}) \underset{t \to 0^+}{\sim} \frac{1}{(4\pi t)^{n/2}} \sum_{k=0}^{\infty} t^k a_k(T_\omega, (D_A - f)^2)$$

(see [6], 1.7).

We first remark that the coefficients $a_k(T_\omega, (D_A - f)^2)$ define 1-forms on $\mathcal{M}(P)$:

Let us denote by $\langle\ ,\ \rangle$ the scalar product on the spaces of p-forms $\Gamma(\mathfrak{g} \otimes \Lambda^p M)$ defined by the Ad-invariant scalar product

$$\langle X, Y\rangle_g := -\frac{1}{2} \text{ Trace } (\rho_*(X) \cdot \rho_*(Y))$$

on the Lie algebra g. $\langle\ ,\ \rangle$ defines a $\mathcal{G}(P)$-invariant weak Riemannian metric on $\mathcal{C}^+(P)$, hence weak metrics on $\mathcal{M}(P)$ and $\mathcal{N}(P)$, respectively. Let $d_A : \Gamma(\mathfrak{g} \otimes \Lambda^p M) \to \Gamma(\mathfrak{g} \otimes \Lambda^{p+1} M)$ denote the covariant differential associated to $A \in \mathcal{C}^+(P)$ and d_A^* its formal adjoint with respect to $\langle\ ,\ \rangle$. Then the tangent space of $\mathcal{C}^+(P)$ at A decomposes into

$$T_A \mathcal{C}^+(P) = \Gamma(\mathfrak{g} \otimes \Lambda^1 M) = Im\ d_A \oplus \text{Ker } d_A^*$$

where $Im\ d_A$ is tha tangent space of the orbit $\mathcal{G}(P) \cdot A \subset \mathcal{C}^+(P)$ and Ker d_A^* is the horizontal lift of the tangent space $T_{\pi(A)}\mathcal{M}(P)$ (see [2], [5]). $\mathcal{G}(P)$ acts in a canonical way on the bundle $S \otimes E$ such that $(D_{\mu A} - f) = \mu(D_A - f)\mu^{-1}$, $\mu \in \mathcal{G}(P)$.

Therefore,

$$\text{Trace}(T_{dl_\mu(\omega)}\, e^{-t(D_{\mu A}-f)^2}) = \text{Trace}(T_\omega\, e^{-t(D_A-f)^2}). \tag{3}$$

Because of (2) and (3) we have for each $k \in \mathbb{N}$ a 1-form $\mathbf{a}_f^{(k)}$ on $\mathcal{M}(P)$ defined by

$$\mathbf{a}_f^{(k)}(\sigma) := a_k(T_{\tilde{\sigma}(A)}, (D_A - f)^2)$$

when $\tilde{\sigma}(A) \in \text{Ker } d_A^*$ denotes the horizontal lift of $\sigma \in T_{\pi(A)}\mathcal{M}(P)$.

1. CALCULATION OF $\mathbf{a}_f^{(0)}, \mathbf{a}_f^{(1)}, \mathbf{a}_f^{(2)}$

Let us denote by $\Omega^A \in \Gamma(\mathfrak{g} \otimes \Lambda^2 M)$ and by $R^A \in \Gamma(\text{End}(E) \otimes \Lambda^2 M)$ the curvature forms associated to $A \in \mathcal{C}^+(P)$, by $\mathcal{R}$ the curvature tensor of M^n and by R the scalar curvature of M^n.

THEOREM 1

(1) $\mathbf{a}_f^{(0)} = 0$

(2) $\mathbf{a}_f^{(1)}(\sigma) = \begin{cases} 0 & n > 3 \\ 4\langle\tilde{\sigma}(A), *\Omega^A\rangle & n = 3 \end{cases}$

(3) $\mathbf{a}_f^{(2)}(\sigma) = -\frac{4}{3}\, 2^{\left[\frac{n}{2}\right]}\langle\tilde{\sigma}(A), d_A^*(f\Omega^A)\rangle + \alpha_n$

$\alpha_n = 0$ if $n \neq 3,5$

$\alpha_3 = -\frac{1}{3}\langle\tilde{\sigma}(A), R *\Omega^A + *(\Omega_{ij}^A \otimes \mathcal{R}_{ij})\rangle$

$$-\frac{2}{3}\sum_{j=1}^{3}\int_M \langle d_A^* d_A(\tilde{\sigma}(A)(s_j)) - \tilde{\sigma}(A)(\Delta s_j), *\Omega^A(s_j)\rangle_{\mathfrak{g}}\, dM$$

$$\alpha_5 = -i\sum_{j=1}^{5}\int_M 2Tr_x(\rho_*(\tilde{\sigma}(A)(s_j))\cdot(*R^A \overset{\circ}{\wedge} R^A(s_j))$$

$$+\frac{1}{6}(*\langle\mathcal{R}\wedge\mathcal{R}\rangle(s_j))\, Tr_x(\rho_*(\tilde{\sigma}(A)(s_j))\, dM$$

where

$$R^A \overset{\circ}{\wedge} R^A(X_1,\ldots,X_4) = \sum_p sgn(p) R^A(X_{p(1)}, X_{p(2)})\cdot R^A(X_{p(3)}, X_{p(4)})$$

$$\langle\mathcal{R}\wedge\mathcal{R}\rangle(X_1,\ldots,X_4) = \sum_p sgn(p)\langle\mathcal{R}(X_{p(1)}, X_{p(2)}), \mathcal{R}(X_{p(3)}, X_{p(4)})\rangle_{\Lambda^2 M}$$

$(s_1,\ldots,s_n)$ *local ON-basis.*

Proof: $(D_A - f)^2$ is a 2^{nd}-order elliptic selfadjoint differential operator with leading symbol $\sigma(D_A - f)^2)_v = +\|v\|^2\, Id$. T_ω is a homomorphism. Therefore we can apply Gilkey's method to calculate the coefficients $a_k(T_\omega, (D_A - f)^2)$. By a straightforward calculation using (1) we obtain a Weitzenböck formula for the operator $(D_A - f)^2$:

$$(D_A - f)^2 = \Delta^{f,A} - \left(-\frac{1}{4}R + (n-1)f^2 - Q_A\right)$$

where $\Delta^{f,A} = \nabla^{f,A\,*}\nabla^{f,A}$ denotes the Bochner-Laplacian for the connection $\nabla^{f,A}: \Gamma(S\otimes E)\to\Gamma(S\otimes E)$

$$\nabla_X^{f,A}(\varphi\otimes e) := \nabla_X^S\varphi\otimes e + \varphi\otimes\nabla_X^A e + f(X\cdot\varphi\otimes e)$$

and Q_A is the homomorphism

$$Q_A(\varphi\otimes e) = \sum_{k<l} s_k\cdot s_l\cdot\varphi\otimes R^A(s_k, s_l)e$$

$(s_1,\ldots,s_n)$ local ON-basis.

According to [6] (1.7, 4.8) we have

$$a_0(T_\omega, (D_A - f)^2) = \int_M Tr_x(T_\omega)\, dM$$

$$a_1(T_\omega, (D_A - f)^2) = \int_M Tr_x(T_\omega \circ H + \frac{1}{6} RT_\omega)\, dM$$

(4)
$$a_2(T_\omega, (D_A - f)^2) = \int_M Tr_x\Big\{T_\omega\Big(-\frac{1}{30}\Delta R + \frac{1}{72}R^2 - \frac{1}{180}\|Ric\|^2 + \frac{1}{180}\|\mathcal{R}\|^2\Big) + Tr_x\Big\{\frac{1}{6}RT_\omega H + \frac{1}{2}T_\omega H^2 + \frac{1}{12}T_\omega\ W_{ij}W_{ij} - \frac{1}{6}T_\omega \Delta^{f,A} H\Big\}dM$$

where H denote the homomorphism $H = -\frac{1}{4}R + (n-1)f^2 - Q_A$, W_{ij} the components of the curvature tensor of the connection $\nabla^{f,A}$ and Ric the Ricci curvature of M^n. Now, consider an ON-basis $e_1, \ldots, e_n \in \mathbb{R}^n \subset \text{Cliff}(\mathbb{R}^n, -\langle,\rangle)$. Then for the Clifford-multiplication we have

$$\text{Trace}(e_i) = 0 \qquad i = 1, \ldots, n$$

$$\text{Trace}(e_i \cdot e_j) = 0 \qquad i < j$$

$$\text{Trace}(e_i \cdot e_j \cdot e_k) = \begin{cases} 0 & n \neq 3 \\ 2 & n = 3 \end{cases} \qquad i < j < k$$

$$\text{Trace}(e_i \cdot e_j \cdot e_k \cdot e_l) = 0 \qquad i < j < k < l$$

$$\text{Trace}(e_i \cdot e_j \cdot e_k \cdot e_l \cdot e_p) = \begin{cases} 0 & n > 5 \\ -4i & n = 5 \end{cases} \qquad i < j < k < l < p.$$

Applying these relations and the definition of the scalar product we obtain

1) $Tr_x(T_\omega) = 0$

2) $Tr_x(T_\omega H) = \begin{cases} 0 & n \neq 3 \\ 4 < \omega,\ {*\Omega^A} > & n = 3 \end{cases}$

$$3)\ Tr_x(T_\omega H^2) = \begin{cases} 0 & n \neq 3,5 \\ -4i \sum_{j=1}^{5} Tr_x(\rho_*(\omega(s_j)) * (R^A \wedge R^A)(s_j)) & n = 5 \\ (-2R + 16 f^2) \langle \omega, *\Omega^A \rangle_x & n = 3 \end{cases}$$

$$4)\ Tr_x(T_\omega W_{ij} W_{ij}) =$$

$$= \begin{cases} 8 \dim S \ \langle \omega, \operatorname{grad} f \lrcorner \Omega^A \rangle_x & n \neq 3,5 \\ 32 \ \langle \omega, \operatorname{grad} f \lrcorner \Omega^A \rangle_x - 2i \sum_{j=1}^{5} * \langle \mathcal{R} \wedge \mathcal{R} \rangle(s_j) Tr_x(\rho_*(\omega(s_j)) & n = 5 \\ 16 \langle \omega, \operatorname{grad} f \lrcorner \Omega^A \rangle_x - 32 f^2 \langle \omega, *\Omega^A \rangle_x - 4 \langle \omega, *(\Omega^A_{ij} \otimes \mathcal{R}_{ij}) \rangle_x & n = 3 \end{cases}$$

$$5) \int_M Tr_x(T_\omega \cdot \Delta^{f,A} H)\, dM = \int_M Tr_x((\Delta^{f,A} T_\omega) \circ H)\, dM =$$

$$= \begin{cases} 8 \dim S \langle \omega, f d_A^* \Omega^A \rangle - 4 \dim S \langle \omega, \operatorname{grad} f \lrcorner \Omega^A \rangle & n \neq 3 \\ 16 \langle \omega, f d_A^* \Omega^A \rangle - 8 \langle \omega, \operatorname{grad} f \lrcorner \Omega^A \rangle + 32 f^2 \langle \langle \omega, *\Omega^A \rangle & n = 3 \\ + 4 \sum_{j=1}^{3} \int_M \langle d_A^* d_A(\omega(s_j)) - \omega(\Delta s_j), (*\Omega^A)(s_j) \rangle_x \, dM. & \end{cases}$$

Using $d_A^*(f\tau) = -\operatorname{grad} f \lrcorner \tau + f d_A^* \tau$, $\tau \in \Gamma(\mathfrak{g} \otimes \Lambda^p M)$, and (4), Theorem 1 follows. ∎

2. SOME PROPERTIES OF $\mathfrak{a}_f^{(j)}$

Consider the «extended» Yang-Mills functional

$$L_f(\pi(A)) = \int_M f(x) |\Omega^A(x)|^2 \ dM \ \text{ on } \mathcal{M}(P).$$

THEOREM 2: *Let $n \geqslant 4, n \neq 5$. Then*

$$\mathfrak{a}_f^{(2)} = -\frac{2}{3} \cdot 2^{\left[\frac{n}{2}\right]} d\, L_f.$$

Proof: Let $A(u) = A + \tau(u)$ be a variation with $\dot{\tau}(0) = \tilde{\sigma}(A)$. Then

$$dL_f(\sigma) = \frac{d}{du}\,(L_f(A + \tau(u)))|_{u=0} = 2 \cdot \int_M f(x) \left\langle \frac{d}{du}\,(\Omega^{A+\tau(u)})|_{u=0}, \Omega^A \right\rangle_x dM.$$

Using $\Omega^{A+\tau} = \Omega^A + d_A \tau + \frac{1}{2}[\tau \wedge \tau]$

Theorem 2 follows from Theorem 1. ∎

Now we derive a relation between the 1-forms $\mathbf{a}_j^{(j)}$ and the η-function of the operator $D_A - f$. Consider the open subset

$$\mathscr{C}_f^+(P) = \{A \in \mathscr{C}^+(P) \mid \mathrm{Ker}(D_A - f) = \{0\}\} \subset \mathscr{C}^+(P).$$

Since the spectrum of $(D_A - f)$ is gauge-invariant $\mathscr{G}(P)$ acts on $\mathscr{C}_f^+(P)$. Let $\mathscr{M}_f(P) := \mathscr{C}_f^+(P)/\mathscr{G}(P)$. Let us denote by $\eta(s, D) = \sum_{\lambda \neq 0} sgn(\lambda) |\lambda|^{-s}$ (λ eigenvalues of D) the η-function of an elliptic selfadjoint operator D of order $d > 0$. $\eta(.\,, D)$ is a meromorphic function on $\mathbb{C}$ with only simple poles which lie in $\left\{\frac{n-k}{d}, k = 0, 1, 2, \ldots\right\}$. In $s = 0, -1, -3, -5, \ldots$ η is regular.

THEOREM 3: *Consider the functions $F_f^{(j)} : \mathscr{M}_f(P) \to \mathbb{R}$:*

1) $F_f^{(0)}(\pi(A)) := -\dfrac{1}{2(n-1)}\,\Gamma\left(\dfrac{n}{2}\right) \mathrm{Res}_{s=n-1}\,\eta(s, D_A - f)$

2) $$F_f^{(1)}(\pi(A)) := \begin{cases} -\frac{1}{2}\sqrt{\pi}\,\eta(0, D_A - f) & n = 3 \\ -\frac{1}{2(n-3)}\,\Gamma\left(\frac{n}{2} - 1\right) \mathrm{Res}_{s=n-3}\,\eta(s, D_A - f) & n \geqslant 3 \end{cases}$$

3) $$F_f^{(2)}(\pi(A)) := \begin{cases} \eta(-1, D_A - f) & n = 4 \\ -\frac{1}{2}\sqrt{\pi}\,\eta(0, D_A - f) & n = 5 \\ -\frac{1}{2(n-5)}\,\Gamma\left(\frac{n}{2} - 2\right) \mathrm{Res}_{s=n-5}\,\eta(5, D_A - f) & n \neq 4,5. \end{cases}$$

Then $F_f^{(j)}$ are correctly defined real functions and

$$\mathbf{a}_f^{(j)} = d\, F_f^{(j)}.$$

Proof. Let $A \in \mathscr{C}_f^+(P)$. Denote by $0 < \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \ldots \to \infty$ the eigenvalues of $(D_A - f)^2$ and by $\varphi_i \in \Gamma(S \otimes E)$ a corresponding ON-basis of eigenfunctions. Then

$$Tr(T_\omega (D_A - f)^2)^{-s}) = \sum_{k=0}^{\infty} \lambda_k^{-s} \langle T_\omega \varphi_k, \varphi_k \rangle_{L^2}$$

$$Re(s) > \frac{n}{2}.$$

By the Melin transform we obtain

$$Tr(T_\omega ((D_A - f)^2)^{-s}) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}\, Tr(T_\omega e^{-t(D_A - f)^2})\, dt$$

$$Re(s) > \frac{n}{2}.$$

The right hand side defines a meromorphic continuation of $Tr(T_\omega ((D_A - f)^2)^{-s})$ to the complex plane with only simple poles in $s_k = \frac{n}{2} - k$, $k = 0, 1, 2, \ldots$, and

$$Res_{s=s_k}\, Tr(T_\omega ((D_A - f)^2)^{-s}) = a_k(T_\omega, (D_A - f)^2) \cdot \Gamma(s_k)^{-1}.$$

In $s = 0, -1, -2, \ldots$, $Tr(T_\omega ((D_A - f)^2)^{-s})$ is regular and has the value

$$Tr(T_\omega ((D_A - f)^2)^{-s})|_{s=-j} = \begin{cases} a_{j+\frac{n}{2}}(T_\omega \cdot (D_A - f)^2)\, Res_{s=-j} \cdot \Gamma(s) & n \text{ even} \\ 0 & n \text{ odd.} \end{cases}$$

(see [6], 1.7, 1.10). Now, let $A(u)$ be a smooth variation of A in $\mathscr{C}_f^+(P)$ with tangent vector ω. Then [3] (Theorem 2.10) implies

$$\frac{d}{du} \eta(z, D_{A(u)} - f)|_{u=0} = -z\, Tr(T_\omega ((D_A - f)^2)^{-\frac{1}{2}(z+1)}).$$

Hence

1. case: $n - 2k - 1 \notin \{0, -1, -3, -5, \ldots\}$

$$Res_{z=n-2k-1} \frac{d}{du} \eta(z, D_{A(u)} - f)|_{u=0} = -2(n-2k-1)$$

$$\cdot a_k (T_\omega, (D_A - f)^2) \Gamma\left(\frac{n}{2} - k\right)^{-1}$$

$$k = 0, 1, 2, \ldots$$

2. case: $n - 2k - 1 \in \{-1, -3, -5, \ldots\}$

$$\frac{d}{du} \eta(n-2k-1, D_{A(u)} - f)|_{u=0} = \begin{cases} a_k (T_\omega (D_A - f)^2) Res_{z=\frac{n}{2}-k} \Gamma(z) & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

3. case: $n - 2k - 1 = 0$

$$\frac{d}{du} \eta(0, D_{A(u)} - f)|_{u=0} = -2a_k (T_\omega, (D_A - f)^2) \sqrt{\pi}^{-1}. \quad \blacksquare$$

Consider a 4-dimensional manifold M with strictly positive scalar curvature R, $R_0 = \min_{x=M} R(x) > 0$.

Let $\mathscr{A}^+_f (P) = \{A \in \mathscr{A}^+ (P) | \operatorname{Ker}(D_A - f) = \{0\}\}$.

PROPOSITION 1: *Let $f \in C^\infty (M^4)$ be a function whose image lies in* $\left(0, \sqrt{\frac{R_0}{4}}\right]$. *Then $\mathscr{A}^+_f (P) = \mathscr{A}^+ (P)$.*

Proof: Let $A \in \mathscr{A}^+ (P)$. The spectrum of D_A is symmetric around zero and the smallest positive eigenvalue $\lambda_1(A)$ is greater than $\sqrt{\frac{R_0}{4}}$ (see [8]). D_A is an essentially selfadjoint operator. Therefore the closure $\overline{D_A - f}$ is a selfadjoint operator in $L^2 (S \otimes E)$ with dense domain and the same spectrum as $D_A - f$.

We decompose

$$\overline{D_A - f} = \overline{D_A - \beta} + \beta - f,$$

where β is a real number. Then $\overline{D_A - \beta}$ is a selfadjoint and $\beta - f$ a bounded symmetric operator in $L^2 (S \otimes E)$. Applying perturbation theory (see [9], V. 4) we obtain

$$(5) \qquad \sup_{z \in spec(D_A - f)} \operatorname{dist}(z, spec(D_A - \beta)) \leq \| \beta - f \| \leq \max_{x \in M} | \beta - f(x)) |$$

Because of $spec(D_A) \cap \left(0, \sqrt{\frac{R_0}{4}}\right] = \emptyset$ we have

$spec(D_A - \beta) \cap \left(-\beta, \sqrt{\frac{R_0}{4}} - \beta\right] = \emptyset$. Let $c_1 := \min_{x \in M} f(x), c_2 = \max_{x \in X} f(x)$.

1. case: If $\sqrt{\frac{R_0}{4}} + c_1 \leqslant 2c_2$, we choose $\beta \in \mathbb{R}$ with

$$\max\left(c_1, \frac{c_2}{2}\right) < \beta < \frac{c_1 + c_2}{2}$$

2. case: If $\sqrt{\frac{R_0}{4}} + c_1 > 2c_2$, we choose $\beta \in \mathbb{R}$ with

$$c_2 < \beta < \frac{1}{2}\left(\sqrt{\frac{R_0}{4}} + c_1\right) .$$

Then it is easy to see that

$$\mathrm{dist}(0, spec(D_A - \beta)) > \max_{x \in M} |\beta - f(x)| .$$

Hence, because of (5), $0 \notin spec(D_A - f)$. ∎

From Theorems 2, 3 and Proposition 1 we obtain

COROLLARY: *Let M be a 4-dimensional Riemannian manifold with strictly positive scalar curvature and $f \in C^\infty(M)$ a function with image in $\left(0, \sqrt{\frac{R_0}{4}}\right]$. Then*

$$\eta(-1, D_A - f) = -\frac{8}{3} \int_M f(x) |\Omega^A|_x^2 \, dM + const$$

for all $A \in \mathscr{A}^+(P)$. ∎

REMARK: Let M be a 4-dimensional Riemannian manifold with strictly positive scalar curvature, $X^5 := M^4 \times S^1$ and $p : X^5 \to M^4$ be the projection. The operator D_{p^*A} on the bundles over X^5 can be considered as operator on a S^1-parametric family of spinors on M^4: $D_{p^*A} = D_A + e_5 \cdot \frac{\partial}{\partial t}$. We can prove $spec(D_{p^*A} - f \cdot p) \cap \left(0, \min\left(1, \sqrt{\frac{R_0}{4}}\right)\right) = \emptyset$ for all $A \in \mathscr{A}^+(P)$. As in

Proposition 1 $\mathrm{Ker}(D_{p*A} - f) = \{0\}$ follows for all $A \in \mathscr{A}^+(P)$ and $f \in C^\infty(M)$ with $\mathrm{Image}(f) \subset \left(0, \min\left(1, \sqrt{\frac{R_0}{4}}\right)\right)$. Applying Theorem 1 we obtain: For each function $f \in C^\infty(M^4)$ with image in $\left(0, \min\left(1, \sqrt{\frac{R_0}{4}}\right)\right)$

$$\eta(0, D_{p*A} - f \cdot p) = \frac{32}{3} \sqrt{\pi} \int_M f(x) |\Omega^A(x)|^2 dM + \text{const}$$

for all $A \in \mathscr{A}^+(P)$.

Now, consider M^n, $n \geqslant 4$, $n \neq 5$ and the «extended» Yang-Mills functional

$$L_f(\pi(A)) = \int_M f(x) |\Omega^A(x)|^2 \, dM.$$

By a straightforward calculation we prove

THEOREM 4: *$\pi(A) \in \mathscr{M}(P)$ is a critical point of $L_f : \mathscr{M}(P) \to \mathbb{R}$ iff $d_A^*(f\Omega^A) = 0$. The Hessian of L_f at a critical point $\pi(A)$ is*

$$HL_f(\omega, \tau) = 2 \langle d_A \tilde{\omega}(A), f d_A \tilde{\tau}(A) \rangle + \\ + 2 \langle [\tilde{\omega}(A) \wedge \tilde{\tau}(A)], f\Omega^A \rangle$$

where $\tilde{\omega}(A) \in \mathrm{Ker}\, d_A^*$ denotes the horizontal lift of $\omega \in T_{\pi(A)} \mathscr{M}(P)$. ■

3. THE 1-FORM $\mathfrak{b}_f$ ON THE INSTANTON SPACE $\mathscr{N}(P)$

Consider a 4-dimensional manifold M and $A \in \mathscr{A}^+(P)$. Let $d_A^- : \Gamma(\mathfrak{g} \otimes \Lambda^1 M) \to \Gamma(\mathfrak{g} \otimes \Lambda^2_- M)$ be the operator $d_A^- := \sqrt{2} p_- d_A$, where p_- is the orthogonal projection on the space $\Gamma(\mathfrak{g} \otimes \Lambda^2_- M)$ of all anti-self-dual 2-forms. Since the complex

(6) $$\Gamma(\mathfrak{g}) \xrightarrow{d_A} \Gamma(\mathfrak{g} \otimes \Lambda^1 M) \xrightarrow{d_A} \Gamma(\mathfrak{g} \otimes \Lambda^2_- M)$$

is elliptic, there is an orthogonal decomposition

$$\Gamma(\mathfrak{g} \otimes \Lambda^1 M) = \mathrm{Ker}\, \Delta_A^1 \oplus Im\, d_A \oplus Im(d_A^-)^*$$

with $\Delta_A^1 := d_A d_A^* + (d_A^-)^* d_A^-$. $\mathrm{Ker}\, d_A^- = \mathrm{Ker}\, \Delta_A^1 \oplus \mathrm{Im}\, d_A$ is the tangent space of $\mathscr{A}^+(P)$ at A. Therefore $\mathrm{Ker}\, \Delta_A^1$ is the horizontal lift of the tangent space $T_{\pi(A)} \mathscr{N}(P)$ (see [2], [5]). Now, let $\mathfrak{b}_f$ be the restriction of the 1-form $\mathfrak{a}_f^{(2)}$ to $\mathscr{N}(P)$. Since $d_A^* \Omega^A = 0$ for all $A \in \mathscr{A}^+(P)$ and $d_A^*(f\Omega^A) = -\mathrm{grad}\, f \lrcorner \Omega^A + f d_A^* \Omega^A$ Theorem 1 implies

$$\mathfrak{b}_f(\omega) = \frac{16}{3} \langle \tilde{\omega}(A), \text{grad } f \lrcorner \Omega^A \rangle$$

where $\tilde{\omega}(A) \in \operatorname{Ker} \Delta_A^1$ is the horizontal lift of $\omega \in T_{\pi(A)} \mathcal{N}(P)$. Furthermore, by Theorem 2

$$\mathfrak{h}_f = -\frac{8}{3} d L_f.$$

Hence $\mathfrak{h}_f$ is the differential of a functional bounded by $\frac{32}{3} \pi^2 k \|f\|_\infty$, where k is the instanton number of P.

A straightforward calculation shows

THEOREM 5: *$\pi(A)$ is a critical point of $L_f : \mathcal{N}(P) \to \mathbb{R}$ iff there exists a 2-form $\Sigma_2 \in \Gamma(\mathfrak{g} \otimes \Lambda_-^2 M)$ such that $\operatorname{grad} f \lrcorner \Omega^A = (d_A^-)^* \Sigma_2$. The Hessian of L_f at a critical point $\pi(A)$ is*

$$H L_f(\omega, \tau) = -2 \langle \tilde{\omega}(A), \operatorname{grad} f \lrcorner d_A \tilde{\tau}(A) \rangle$$
$$- 2\sqrt{2} \langle p_-[\tilde{\omega}(A) \wedge \tilde{\tau}(A)], \Sigma_2 \rangle$$

where $\tilde{\omega}(A), \tilde{\tau}(A) \in \operatorname{Ker} \Delta_A^1$ are the horizontal lifts of ω, τ. ∎

Now, consider the vector field $X_f(A) = -d_A^*(f\Omega^A)$ on $\mathscr{C}^+(P)$. Then $X_f(A) = \operatorname{grad} f \lrcorner \Omega^A$ for $A \in \mathscr{A}^+(P)$. X_f is left-invariant with respect to the $\mathscr{G}(P)$-action on $\mathscr{C}^+(P)$. It is easy to check that X_f is a horizontal vector field. Therefore X_f defines a vector field $Z_f := d\pi(X_f)$ on $\mathscr{M}(P)$.

PROPOSITION 2: *Let $f \in C^\infty(M^4)$ be a function such that* $\operatorname{grad} f \neq 0$ dM^4*-almost everywhere.*

Then for all non-flat selfdual connections A

$$X_f(A) = \operatorname{grad} f \lrcorner \Omega^A \neq 0.$$

Proof: Assume $X_f(A) = 0$. For $x \in M^4$ such that $\operatorname{grad} f(x) \neq 0$ we can choose an orthonormal basis $e_1 = \frac{\operatorname{grad} f}{\|\operatorname{grad} f\|}, e_2, e_3, e_4$ around x. Then $X_f(A)(x) = \|\operatorname{grad} f(x)\| \, \Omega^A_{1k}(x) \, e_k^*(x)$ is zero iff $\Omega^A_{1k}(x) = 0$ for $k = 2, 3, 4$. From the self-duality of A we obtain $\Omega^A(x) = 0$. This holds dM^4-almost everywhere, hence A is flat. ∎

Under the assumption of Proposition 2 the vector field Z_f does not vanish on $\mathcal{N}(P)$. We now prove a condition under which Z_f is tangent to $\mathcal{N}(P)$.

PROPOSITION 3: *Let $f \in C^\infty(M^4)$ be a function such that the space $\Gamma(\Lambda_+^2 M)$ of all self-dual 2-forms is invariant under the Lie derivative $\mathscr{L}_{\operatorname{grad} f}$:*

$$\mathscr{L}_{\mathrm{grad}\, f}(\Gamma(\Lambda^2_+ M)) \subset \Gamma(\Lambda^2_+ M).$$

Then for $A \in \mathscr{A}^+(P)$

$$\Delta^1_A\ (\mathrm{grad} f \lrcorner \Omega^A) = 0.$$

Proof: Let $Y := \mathrm{grad}\, f$. $Y \lrcorner \Omega^A \in \mathrm{Ker}\, \Delta^1_A$ iff $Y \lrcorner \Omega^A \in \mathrm{Ker}\, d^*_A$ and $Y \lrcorner \Omega^A \in$ $\in \mathrm{Ker}\, d^-_A$.

1) Because of $d^*_A\ \Omega^A = 0$ we have

$$\langle d_A s,\ Y \lrcorner \Omega^A \rangle = \langle d_A s, -d^*_A (f\Omega^A)\rangle = -\langle d_A d_A s, f\,\Omega^A \rangle =$$

$$= -\langle [\Omega^A, s], f\,\Omega^A \rangle = \sum_{j<k} \int_M \langle fs,\ [\Omega^A_{jk}, \Omega^A_{jk}]\rangle\, dM$$

$$= 0.$$

2) Choose a local ON-basis f^1, f^2, f^3 in $\Gamma(\Lambda^2_+ M)$.
Then locally $\Omega^A = \omega_i \otimes f^i$ and

$$d_A (Y \lrcorner \Omega^A) = d_A (\omega_i \otimes Y \lrcorner f^i) = d_A\, \omega_i \wedge (Y \lrcorner f^i) - \omega_i \otimes d(Y \lrcorner f_i)$$

$$= -\omega_i \otimes \mathscr{L}_Y(f_i) + \omega_i \otimes Y \lrcorner df_i + d_A\, \omega_i \wedge (Y \lrcorner f^i)$$

$$= -\omega_i \otimes \mathscr{L}_Y(f_i) - (d_A\, \omega^i)(Y) \otimes f^i + Y \lrcorner d_A \Omega^A.$$

Using the Bianci Identity we obtain from $\mathscr{L}_Y(f^i) \subset \Gamma(\Lambda^2_+ M)$ $d^-_A (Y \lrcorner \Omega^A) = 0$. ■

We now consider the case $S^4 = M^4$.
Let $f_v \in C^\infty(S^4)$ be the function

$$f_v(x) := \langle v, x\rangle,\ v \in \mathbb{R}^5,\ v \neq 0.$$

$\{f_v \mid v \in \mathbb{R}^5,\ v \neq 0\}$ is the space of all eigenfunctions of the smallest positive eigenvalue of the Laplacian on S^4.

It is easy to prove that $\mathrm{grad}\, f_v(x) = v - \langle x, v\rangle x$ is an infinitesimal conformal transformation on S^4. Hence by Proposition 2 and 3 the vector field Z_{f_v} is a nowhere vanishing tangent vector field on $\mathscr{N}(P)$. Because of $\mathfrak{h}_{f_{\alpha v + \beta w}} = \alpha\ _{f_v} +$ $+ \beta \mathfrak{h}_{f_w}$ and $\mathfrak{h}_{f_v}(Z_{f_v}(\pi(A))) = \dfrac{16}{3} \mid X_f(A) \mid^2 > 0$ we obtain

THEOREM 6: *Let $(v_1, \ldots, v_5) \in \mathbb{R}^5$ be a basis of $\mathbb{R}^5$. Then $\mathfrak{h}_{f_{v_1}}, \ldots, \mathfrak{h}_{f_{v_5}}$ are closed 1-forms on $\mathscr{N}(P)$ which are linearly independent at each point of $\mathscr{N}(P)$. Hence $\{\mathfrak{h}_{f_{v_1}}, \ldots, \mathfrak{h}_{f_{v_5}}\}$ determines a foliation of codimension 5 on $\mathscr{N}(P)$.*

The normal space to a leaf of this foliation at a point $\pi(A)$ is

$$\text{span}\{Z_{f_v}(\pi(A)) = d\pi(\text{grad} f_v \lrcorner \Omega^A) \mid v \in \mathbb{R}^5\}. \qquad \blacksquare$$

THEOREM 7:

1) *The second fundamental form of a leaf $\mathscr{F} \subset \mathscr{N}(P)$ at $\pi(A)$ is* $\alpha_{Z_f}(X, Y) = \langle f d_A \widetilde{X}(A), d_A \widetilde{Y}(A)\rangle + \langle [\widetilde{X}(A) \wedge \widetilde{Y}(A)],\ f\Omega^A\rangle$

$$X, Y \in T_{\pi(A)} \mathscr{F}.$$

2) *For the commutator of two normal fields of the foliation we have*

$$\widetilde{[Z_{f_v}, Z_{f_w}]}(A) = -f_v X_{f_w}(A) + f_w X_{f_v}(A) + 2\, d_A G_A^0 (\Omega^A(\text{grad} f_v, \text{grad} f_w))$$

where G_A^0 is the Green operator to the Laplacian $\Delta_A^0 = d_A^* d_A$ of the elliptic complex (6).

Proof: The Levi-Civita connection on $\mathscr{N}(P)$ is given by

$$\nabla_X^{\mathscr{N}} Z_{\pi(A)} = proj_{T\mathscr{N}}\, d\pi(d\widetilde{Z}(\widetilde{X}(A))) \tag{7}$$

Therefore,

$$\begin{aligned}
\alpha_{Z_f}(X, Y) &= \langle \nabla_X^{\mathscr{N}} Y, Z_f\rangle = -\langle \nabla_X^{\mathscr{N}} Z_f, Y\rangle = -\langle dX_f(\widetilde{X}(A)), \widetilde{Y}(A)\rangle = \\
&= \frac{d}{dt}(d^*_{A+t\widetilde{X}}(f\Omega^{A+t\widetilde{X}}))\big|_{t=0}, \widetilde{Y}(A)\rangle = \\
&= \langle d_A^*(f d_A \widetilde{X}(A)) - f{*}[\widetilde{X}(A) \wedge \Omega^A], \widetilde{Y}(A)\rangle \\
&= \langle f d_A \widetilde{X}(A), d_A \widetilde{Y}(A)\rangle + \langle [\widetilde{X}(A) \wedge \widetilde{Y}(A)],\ f\Omega^A\rangle.
\end{aligned}$$

From (7) we obtain

$$\begin{aligned}
\widetilde{[Z_f, Z_g]}(A) &= proj_{\text{Ker}\Delta_A^1}(dX_g(X_f) - dX_f(X_g)) = \\
&= proj_{\text{Ker}\Delta_A^1}(-d_A^*(g d_A X_f) + d_A^*(f d_A X_g) + g{*}[X_f(A) \wedge \Omega^A] \\
&\quad - f{*}[X_g(A) \wedge \Omega^A].
\end{aligned}$$

Since $X_f(A) \in \text{Ker}\, d_A^-$ we have $d_A X_f(A) \in \Gamma(\mathfrak{g} \otimes \Lambda_+^2 M)$ and $d_A^* d_A X_f(A) = {*}\,[X_f(A) \wedge \Omega^A]$. Therefore,

$$\begin{aligned}
\widetilde{[Z_f, Z_g]}(A) &= proj_{\text{Ker}\Delta_A^1}(\text{grad}\, g \lrcorner d_A(X_f(A) - \text{grad} f \lrcorner d_A(X_g(A))) \\
&= proj_{\text{Ker}\Delta_A^1}(d_A(\Omega^A(\text{grad}\, g, \text{grad} f)) + [\text{grad}\, g, \text{grad} f] \lrcorner \Omega^A).
\end{aligned}$$

Because of $[\text{grad}\, g, \text{grad} f] = g \cdot \text{grad} f - f \cdot \text{grad}\, g$ for linear functions g and f on S^4 we obtain

$$[\widetilde{Z_f, Z_g}](A) = proj_{\mathrm{Ker}\,\Delta^1_A}(g \cdot X_f(A) - f \cdot X_g(A)).$$

Now, using $g\,X_f(A) - f\,X_g(A) \in \mathrm{Ker}\, d_A^-$ we get

$$\begin{aligned}[\widetilde{Z_f, Z_g}](A) &= g\,X_f(A) - f\,X_g(A) - d_A\,G_A^0\,d_A^*(gX_f(A) - fX_g(A)) \\ &= g\,X_f(A) - f\,X_g(A) + 2\,d_A\,G_A^0\,(\Omega^A\,(\mathrm{grad}\,f, \mathrm{grad}\,g)).\end{aligned}$$ ■

Now, let us consider the special case of $SU(2)$-instantons on S^4. The $SU(2)$-principal bundles on S^4 are classified by $k \in \mathbb{Z}$ and for each bundle P_k we have $\dim \mathcal{N}(P_k) = 8k - 3$ ($k \geqslant 1$) (see [2]). Hence the leaves of the foliation in $\mathcal{N}(P_k)$ have dimension $8(k-1)$.

The minimum of the Yang-Mills functional $L(A) = \int_{S^4} |\,\Omega^4\,|^2\, dS^4$ on $\mathcal{M}(P_k)$ is $4\pi^2 k$: $4\pi^2 k = \int_{S^4} |\,\Omega^4\,|^2\, dS^4$ for all $A \in \mathcal{N}(P_k)$. Let $e_1, \dots, e_5 \in \mathbb{R}^5$ be the canonical basis of $\mathbb{R}^5$. Then

$$\mathrm{L}_k : \mathcal{N}(P_k) \to B(0, 4\pi^2 k) \subset \mathbb{R}^5$$

$$\pi(A) \to (L_{f_{e_1}}(A), \dots, L_{f_{e_5}}(A)) = \int_{S^4} \vec{x}\,|\,\Omega^A(x)\,|^2\, dS^4$$

maps the instanton space $\mathcal{N}(P_k)$ to an open ball of $\mathbb{R}^5$ with radius $4\pi^2 k$.

From Theorem 6 we know that $d\mathrm{L}_k$ ($k \geqslant 2$) is surjective and $d\mathrm{L}_1$ is injective. Hence L_k ($k \geqslant 2$) is a submersion and L_1 an immersion in $B(0, 4\pi^2 k)$. We show that L_1 is bijective. We use the description of $\mathcal{N}(P_1)$ derived in [7]. The $SU(2) = Sp(1)$-principal bundle on S^4 is constructed as follows: Let $\mathbb{H}$ be the skew-field of quaternions and $\{a_i,\ i = 1, 2, 3\}$ a basis of the imaginary quaternions with $a_i^2 = -1$ and $a_1 \cdot a_2 = a_3$. Furthermore, let $\varphi : S^4 \to \mathbb{H} \cup \{\infty\}$ be the stereographic projection such that for the standard basis $\{e_1, \dots, e_5\}$ of $\mathbb{R}^5$ $\varphi(e_1) = 1$, $\varphi(e_{i+1}) = a_i$ $i = 1, 2, 3$, $\varphi(-e_5) = 0$. On $\mathbb{H} \cup \{\infty\}$ the group $Sp(2)$ acts transitively by $g \cdot x = (g_{11}x + g_{12}) \cdot (g_{21}x + g_{22})^{-1}$ for $g = (g_{ij}) \in Sp(2)$, $x \in \mathbb{H}$. $Sp(1) \times Sp(1)$ is the isotropy group of $0 \in \mathbb{H}$. The homomorphism $\nu : Sp(2) \to SO(5)$ defined by $\nu(g)y = \varphi^{-1}(g \cdot \varphi(y))$ for $g \in Sp(2)$ and $y \in S^4$ is a double covering with $\nu(Sp(1) \times Sp(1)) = SO(4)$. Hence the principal bundle $(Sp(2), \sigma, S^4; Sp(1) \times Sp(1))$ with projection $\sigma : g \in Sp(2) \to \nu(g)(-e_5) \in S^4$ is the spinor structures of S^4. The associated bundle $P_1 := Sp(2) x_{pr_1} Sp(1)$, where $pr_1 : Sp(1) \times Sp(1) \to Sp(1)$ denotes the projection onto the first component, is the principal $Sp(1)$-bundle over S^4 with index $k = 1$. (see [1]). On the Lie algebra $sp(2)$ we fix the inner product defined by $\langle X, Y \rangle = 2\,Re\,Tr(X\overline{Y}^T)$.

We denote by A_1 the connection on P_1 induced by the canonical $Sp(2)$-invariant connection of the spinor structure of S^4. A_1 is an irreducible self-dual connection which is called the basic instanton. Furthermore, let us denote by $A_t \in \mathscr{A}^+(P_1)$ the connection defined by $h_t^* A_1$, where $h_t : P_1 \to P_1$ is the lift of the conformal map $x \in S^4 \to \varphi^{-1}(t^{-(1/2)} \varphi(x)) \in S^4$. $Sp(2)$ acts canonically on P_1, therefore

$$Sp(2) \times \mathscr{N}(P_1) \to \mathscr{N}(P_1)$$
$$(g, \pi(A)) \to \pi(l_{g^{-1}}{}^* A)$$

defines a left-action of $Sp(2)$ on $\mathscr{N}(P_1)$. In [7] it has been proved that

1) $\mathscr{N}(P_1)$ is the disjoint union of the orbits $Sp(2) \cdot \pi(A_t)$ where $t \in [1, \infty)$.

2) $Sp(2) \cdot \pi(A_1) = \pi(A_1)$ and $Sp(2) \cdot \pi(A_t)$ is isometric to a sphere S_t^4, $t \in \in (1, \infty)$.

3) In the parametrization $\mathscr{N}(P_1) \setminus \pi(A_1) = \bigcup_{t \in (1,\infty)} S_t^4 = (1, \infty) \times S^4$ the Riemannian metric of $\mathscr{N}(P_1)$ is given by

$$g_{\mathscr{N}1} = h(t)^2\, dt^2 + r(t)^2\, g_{S^4}$$

with

$$h(t)^2 = 4\pi^2 \left(\frac{t^2 + 10t + 1}{t(t-1)^4} - \frac{6(t+1)}{(t-1)^5} \ln t \right)$$

$$r(t)^2 = 2\pi^2 \left(\frac{t^2 - 8t + 1}{(t-1)^2} + \frac{12t^2}{(t+1)(t-1)^3} \ln t \right).$$

PROPOSITION 4: *Let $f \in C^\infty(S^4)$ be a linear function. Then for $L_f : \mathscr{N}(P_1) \to \mathbb{R}$ we have*

$$L_f(t, y) = 4\pi^2 f(y)\, a(t)$$

where

$$a(t) = \frac{t^4 - 8t^3 + 8t - 1 + 12t^2 \cdot \ln t}{(t-1)^4}.$$

Proof: Let $g \in Sp(2)$ and $G = \nu(g) \in So(5)$. Then

$$L_f(g \cdot \pi(A_t)) = \int_{S^4} f(y) |\, \Omega^{g \cdot A_t}(y) |^2\, dS^4 =$$

$$= \int_{S^4} f(y) | \ell^*_{g^{-1}} \Omega^{A_t}(y)|^2 \, dS^4$$

$$= \int_{S^4} f(G \cdot y) | \Omega^{A_t}(y)|^2 \, dS^4 .$$

Choose a *ON*-basis on $S^4 \setminus \{e_5\} \simeq \mathbb{H}$ by $s_j(x) = 1/2 \, (1 + \| x \|^2) \, \partial_j x, \, j = 1, \ldots, 4, \, x = x_1 + x_2 a_1 + x_3 a_2 + x_4 a_3 \in \mathbb{H}$. Then for the local connection forms with respect to the section in $P_1 |_{\mathbb{H}}$ defined by s we have

$$A_t(x) = \frac{1}{(t + \| x \|^2)} \, \mathrm{Im}(x \, d\bar{x}) \quad \text{and}$$

$$\Omega^{A_t}(x) = \frac{t}{(t + \| x \|^2)^2} \, dx \wedge d\bar{x}$$

(see [7], [1]).

Using $\varphi^{-1}(x) = \dfrac{1}{1 + \| x \|^2} \, (2x_1, \ldots, 2x_4, \| x \|^2 - 1)$ we obtain

$$L_f(g \cdot \pi(A_t)) = 24 \, t^2 \, f([G]_5) \cdot \int_{\mathbb{R}^4} \frac{\| x \|^2 - 1}{(t + \| x \|^2)^4 (\| x \|^2 + 1)} \, dx$$

where $[G]_5$ denotes the 5. column of G. We identify S^4_t with $Sp(2) \cdot \pi(A_t) \subset \subset \mathcal{N}(P_1)$ such that $\pi(A_t)$ corresponds to the north pole of S^4_t. Then $[G]_5 = y$. Calculating the integral, yields the proposition. ∎

$a(t)$ is a strictly increasing function with

$$\lim_{t \to 1} a(t) = 0 \quad \text{and} \quad \lim_{t \to \infty} a(t) = 1 .$$

COROLLARY: *The map* $\mathrm{L}_1 : \mathcal{N}(P_1) \to \mathrm{B}(0, 4\pi^2) \subset \mathbb{R}^5$ *is given by*

$$\mathrm{L}_1 (t, y) = 4\pi^2 y \, a(t).$$

L_1 *is an embedding of* $\mathcal{N}(P_1)$ *in* $\mathbb{R}^5$ *with image* $B(0, 4\pi^2)$. ∎

Finally, we consider another example.

Take the function $f(y) = y_5^2$ on S^4. By a similar calculation as above we obtain

PROPOSITION 5: *For* $L_f : \mathcal{N}(P_1) \to \mathbb{R}$ *we have*

$$L_f(t, y) = 4\pi^2 \{y_5^2 + (1 - 5y_5^2)\, b(t)\}$$

where

$$b(t) = \frac{2}{(t-1)^5}\,(t^4 + 9t^3 - 9t^2 - t - 6(t^3 + t^2)\cdot \ln t) \qquad ■$$

$b(t)$ is a strictly decreasing function with

$$\lim_{t\to\infty} b(t) = 0 \text{ and } \lim_{t\to 1} b(t) = \frac{1}{5}\,.$$

Hence $L_f|_{S_t^4}$ attains its maximum at the north and south pole and its minimum on the equator of S_t^4. $L_f|_{(1,\infty)\times\{t\}}$ is strictly increasing iff $y_5^2 > 1/5$, constant iff $y_5^2 = 1/5$ and strictly decreasing iff $y_5^2 < 1/5$. In $\pi(A_1)$ L_f has a maximum along the coordinate lines $ty_1, \ldots, ty_4$ and a minimum along the coordinate line ty_5. This proves

PROPOSITION 6: *Let $f(y) = y_5^2$. Then $L_f : \mathcal{N}(P_1) \to \mathbb{R}$ has exactly one critical point, the basic instanton of $\mathcal{N}(P_1)$. This point is non-degenerated with index 4. L_f extends to the compactification $\overline{\mathcal{N}(P_1)} = \pi(A_1) \cup (1, \infty] \times S^4$ and attains its maximum $4\pi^2$ at the north and south pole $(\infty)\times\{\pm e_5\}$ and its minimum zero at the equator $\{\infty\}\times\{x \in S^4 \mid y_5 = 0\}$.* ■

Remark: If we compactify $\mathcal{N}(P_1)$ by S^4: $\overline{\mathcal{N}(P_1)} = [A_1] \cup (1. \infty] \times S^4$, then in the examples of Propositions 4 and 5 $L_f(\infty, y) = 4\pi^2 f(y)$. This reflects the general fact that for a simply connected 4-manifold of positive definite intersection form the $SU(2)$-instantons of index 1 form a 5-dimensional manifold which can be compactified by adding X in such a way that a point x of X^4 represents the limit point of a sequence of connections whose curvatures concentrated in diminishing balls around x (see [4]). Hence in general $L_f : \overline{\mathcal{N}(P_1; X)} \to \mathbb{R}$ satisfies $L_f|_X = 4\pi^2 f$ on $X = \partial\overline{\mathcal{N}(P_1, X)}$.

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Manuscript received: March 11, 1988